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Letters to the Editor

A Note on Rational Chebyshev Approximation on the Positive Real Axis

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We prove here the following

THEOREM. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k \geqslant 0$ $(k \geqslant 0)$, be any entire function of order $\rho(0 < \rho < \infty)$, type τ and lower type $\omega(0 < \omega \leqslant \tau < \infty)$. Then one can not find algebraic polynomials $P_n(x)$ and $Q_n(x)$ with nonnegative real coefficients and of degree at most n for which*

$$\varliminf_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{P_n(x)}{Q_n(x)} \right\|_{L_\infty[0,\infty[} \right\}^{\frac{1}{n}} < (2\sqrt{2})^{\frac{-\tau}{\rho\omega}}. \tag{1}$$

Proof. We show here that we always have, for all large n,

$$\left\| \frac{1}{f(x)} - \frac{P_n(x)}{Q_n(x)} \right\|_{L_\infty[0,\infty]} > (2\sqrt{2})^{\frac{-n\tau}{\rho\omega}}. \tag{2}$$

Otherwise, for infinitely many n,

$$\left\| \frac{1}{f(x)} - \frac{P_n(x)}{Q_n(x)} \right\|_{L_\infty[0,\infty]} \leqslant (2\sqrt{2})^{\frac{-n\tau}{\rho\omega}}. \tag{3}$$

By definition,

$$\frac{\varlimsup}{\varliminf}_{r\to\infty} \frac{\log M(r)}{r^\rho} = \frac{\tau}{\omega} \tag{4}$$

where $M(r) \equiv \mathrm{Max}_{|z|=r} |f(z)|$.

Given any $\epsilon > 0$, there exists an $r_0 = r_0(\epsilon)$ such that for all $r \geqslant r_0(\epsilon)$,

$$\omega(1-\epsilon)\, r^\rho < \log M(r) < \tau(1+\epsilon)\, r^\rho. \tag{5}$$

From (5) it is easy to see that

$$\log M\left(r\left(\frac{1000\tau}{931\omega}\right)^{\frac{1}{\rho}}\right) > \tau(1-\epsilon)\left(\frac{1000}{931}\right) r^\rho > \left(\frac{1000}{931}\right)\left(\frac{1-\epsilon}{1+\epsilon}\right) \log M(r).$$

Hence,

$$M\left(r\left(\frac{1000\tau}{931\omega}\right)^{\frac{1}{\rho}}\right) > \{M(r)\}^{\frac{1000(1-\epsilon)}{931(1+\epsilon)}}. \tag{6}$$

For every $n \geqslant$ some n_1, we can find an r such that

$$f(r) = (2\sqrt{2})^{\frac{n931\tau}{\rho(1000)\omega}}. \tag{7}$$

For this r, if n is sufficiently large, we have

$$\frac{Q_n(r)}{P_n(r)} < (2\sqrt{2})^{\frac{n(931.2)\tau}{\rho(1000)\omega}}, \tag{8}$$

because otherwise, as one easily sees, (3) will be contradicted. But because of (6) and (7),

$$f\left(r\left(\frac{1000\tau}{931\omega}\right)^{\frac{1}{\rho}}\right) > \{f(r)\}^{\frac{1000(1-\epsilon)}{931(1+\epsilon)}} = (2\sqrt{2})^{\frac{n\tau(1-\epsilon)}{\rho\omega(1+\epsilon)}}. \tag{9}$$

Since the coefficients of $P_n(x)$ and $Q_n(x)$ are nonnegative,

$$Q_n\left(r\left(\frac{1000\tau}{931\omega}\right)^{\frac{1}{\rho}}\right)\Big/P_n\left(r\left(\frac{1000\tau}{931\omega}\right)^{\frac{1}{\rho}}\right) < \left(\frac{1000\tau}{931\omega}\right)^{\frac{n}{\rho}} \frac{Q_n(r)}{P_n(r)}$$
$$< \left(\frac{1000\tau}{931\omega}\right)^{\frac{n}{\rho}} (2\sqrt{2})^{\frac{(931.2)n\tau}{1000\omega\rho}}. \tag{10}$$

Now, it is easy to verify that, for $x = r(1000\tau/931\omega)^{1/\rho}$, and $\epsilon > 0$ sufficiently small,

$$(2\sqrt{2})^{-\frac{n\tau}{\rho\omega}} < (\omega 931)^{\frac{n}{\rho}} (\tau 1000)^{-\frac{n}{\rho}} (2\sqrt{2})^{-\frac{(931.2)n\tau}{1000\omega\rho}} - (2\sqrt{2})^{-\frac{n\tau(1-\epsilon)}{\rho\omega(1+\epsilon)}}$$
$$< \frac{P_n(x)}{Q_n(x)} - \frac{1}{f(x)},$$

contradicting (3).

REFERENCE

1. R. P. BOAS, "Entire functions," Academic Press, New York, 1954.

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